

COMPOSITE AND INVERSE OF MULTIVARIATE FUNCTIONS AND ALGEBRAIC SYSTEM OF EQUATIONS

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Abstract

In this paper we extended composite and inverse to functions of many variables. As an application we gave formula solutions expressed by multivariate functions to general transcendental equations. We built algebraic system of equations. The equations contained in it will be algebraic equations if the elements of its basic set are numbers and the equations will be operator equations if the elements of its basic set are functions. Actually we will have a united concept for algebraic equations and operator equations. We discussed the probability to express these solutions in binary functions or unary functions.

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1. Introduction

We will not give any reference about the composite and inverse of multivariate functions as the concepts for unary functions are very familiar for all of us and the concepts for multivariate functions are never stated before. One think the problem about general transcendental equations is closed after Abel's having given the conclusion that there is no solution by radicals for a general polynomial algebraic equation if $n \geq 5$. This is a big mistake. Please reference 'Discrete Algebraic Equations and Discrete Operator

Equations'[6] or 'Representation and Superposition of Discrete Functions and Equations with Parameterized Operations' [7] for the history about solution by radicals for a general polynomial algebraic equation. It has been a short communication in ICM2010, Hyderabad[8]. This paper involves the Hilbert 13th problem. We supply references [1][2][3][4][5] about it.

2. Composite and Inverse of multivariate functions

It is import to know how many variables in a function for us because we will use these functions so frequently in our paper. Superscript with wavy line will be used to indicate the number of variables for a function and it will be omitted for a binary function.

For clearer expression we give new symbolics for several known operations, φ_a for addition, φ_m for multiplication, φ_d for division, φ_p for power, φ_r for root and φ_l for logarithm.

2.1 composite for multivariate functions

We will define composite between functions with same number of variables, two functions of n variables, $\varphi_1^{\tilde{n}}$ and $\varphi_2^{\tilde{n}}$. We should take a function of m ($m < n$) variables $\varphi^{\tilde{m}}$ as a special one of n variables $\varphi^{\tilde{n}}$. Each variable of $\varphi^{\tilde{m}}$ must be indicated to be which variable of $\varphi^{\tilde{n}}$ because $m < n$. Such as different special functions of 4 variables $\varphi_1^{\tilde{4}} = \varphi_p(x_1, x_4) = x_1^{x_4}$ and $\varphi_2^{\tilde{4}} = \varphi_p(x_4, x_2) = x_4^{x_2}$ are obtained by taking the same variable of φ_p as the different variable of $\varphi^{\tilde{4}}$ so we introduce the concept of variables identify.

Definition 2.1 Variables identify Let $\varphi^{\tilde{m}}(v_1, \dots, v_j, \dots, v_m)$ is a function of m variables. $\varphi^{\tilde{m}}$ will be changed to a function of n ($m < n$) variables $\varphi^{\tilde{n}}$ by taking j -th variable of $\varphi^{\tilde{m}}$ as the i_j -th variable of $\varphi^{\tilde{n}}$ like:

$$w^{\tilde{n}}(u_1, u_2, \dots, u_n) = [A_{i_1, \dots, i_j, \dots, i_m}^n(w^{\tilde{m}})](u_1, u_2, \dots, u_n) = w^{\tilde{m}}(i_1, \dots, i_j, \dots, i_m) \quad (1)$$

There are $n \times (n-1) \times \dots \times (n-m+1)$ combinations for a function of m variables $\varphi^{\tilde{m}}$ that will be changed to a function of n variables $\varphi^{\tilde{n}}$.

For example:

$$w_1^{\tilde{4}}(x_1, x_2, x_3, x_4) = x_1 + x_2 \iff w_1^{\tilde{4}} = A_{1,2}^4(\varphi_a)$$

$$w_2^{\tilde{4}}(x_1, x_2, x_3, x_4) = x_3 + x_4 \iff w_2^{\tilde{4}} = A_{3,4}^4(\varphi_a)$$

$$w_3^{\tilde{4}}(x_1, x_2, x_3, x_4) = x_1 + x_4 \iff w_3^{\tilde{4}} = A_{1,4}^4(\varphi_a)$$

The same binary function φ_a is changed to different functions of 4 variables, $w_1^{\tilde{4}}, w_2^{\tilde{4}}$ and $w_3^{\tilde{4}}$ by different $A_{1,2}^4, A_{3,4}^4$ and $A_{1,4}^4$ respectively.

Definition 2.2 composite for multivariate functions composite between two functions of n variables $\varphi_1^{\tilde{n}}$ and $\varphi_2^{\tilde{n}}$ is defined like:

$$\varphi_3^{\tilde{n}} = \varphi_1^{\tilde{n}}[x_1, \dots, x_{i-1}, \varphi_2^{\tilde{n}}(x_1, \dots, x_n), x_{i+1}, \dots, x_n] \quad (2)$$

We introduce operator C_i^m to express the relation between $\varphi_3^{\tilde{n}}$ and $\varphi_1^{\tilde{n}}, \varphi_2^{\tilde{n}}$. The subscript i of C_i^m means replacing i-th variable x_i of $\varphi_1^{\tilde{n}}$ by $\varphi_2^{\tilde{n}}$.

$$\varphi_3^{\tilde{n}} = \varphi_1^{\tilde{n}} C_i^m \varphi_2^{\tilde{n}} \quad (3)$$

Note:

1 $\varphi_1^{\tilde{n}}$ or $\varphi_2^{\tilde{n}}$ may be functions with symbolic of variables like $x_1 + x_4$ or functions with no symbolic of variables like $A_{1,2}^4(\varphi_a)$.

2 $\varphi_1^{\tilde{n}}$ or $\varphi_2^{\tilde{n}}$ may be special function of n variables lacking several variables. For example, $(x_0^2 + x_2^3)C_0^4(x_1^2 + x_3^4) = (x_1^2 + x_3^4)^2 + x_2^3$, there is no x_1, x_3 in $\varphi_1^{\tilde{n}}$ and there is no x_0, x_2 in $\varphi_2^{\tilde{n}}$.

3 Replacement will not happen if $\varphi_1^{\tilde{n}}$ lacks the i-th variable which is defined by the subscript of C_i^m . In this situation the result of composite will be $\varphi_1^{\tilde{n}}$, such as $aC_1^3(x_1^2 + x_3^4) = a$.

4 The result of composite will be a function of n-1 variables if $\varphi_2^{\tilde{n}}$ is a constant like $(x_0 + x_1^2 + x_2^3)C_2^3a = x_0 + a^2 + x_2^3$.

Special situation:

if $n \geq 2, \varphi_2^{\tilde{n}} = x_j, i \neq j$, then each x_i in $\varphi_1^{\tilde{n}}$ will be replaced by x_j then we obtain a function of n-1 variables.

$$\psi^{\tilde{n}-\bar{1}} = \varphi_1^{\tilde{n}} C_i^n x_j = \varphi_1^{\tilde{n}} C_i^n [A_j^n(e)]$$

Here e is the identity function. $\psi^{\tilde{n}-\bar{1}}$ is also called oblique projection of $\varphi_1^{\tilde{n}}$ about i-th variable and j-th variable. The oblique projection of $\varphi_1^{\tilde{n}}$ is depend on only $\varphi_1^{\tilde{n}}, x_i$ and x_j so we give it another expression:

$$\psi^{\tilde{n}-\bar{1}} = C_{i,j}^m(\varphi_1^{\tilde{n}}) \quad (4)$$

For example: $x^x = f^{\bar{1}}(x) = [C_{1,2}^2(\varphi_p)](x)$

By the oblique projection $C_{1,2}^2$ we change the binary function φ_p to a unary function $C_{1,2}^2(\varphi_p)$ and we reduce the number of 'x' from two to one and this is the thing we want to do when we solve equations.

The double branches expression, $w^{\bar{4}}(u_1, u_2, u_3, u_4) = \varphi_3[\varphi_1(u_1, u_2), \varphi_2(u_3, u_4)] = (u_1\varphi_1u_2)\varphi_3(u_3\varphi_2u_4)$ is obtained by replacing two variables of φ_3 by φ_1 and φ_2 respectively. Thus $w^{\bar{4}}$ can be written in $\left(\{ [A_{1,3}^4(\varphi_3)] C_1^4 [A_{1,2}^4(\varphi_1)] \} C_3^4 [A_{3,4}^4(\varphi_2)] \right)$.

It is also called the structural expression of w^4 because this expression contains all structural elements of w^4 . Note,subscripts of $A_{1,3}^4$ is not 1,2 but 1,3. The expression will mean $\varphi_3\{\varphi_1[x, \varphi_2(x, b)], \varphi_2(x, b)\}$ if we replace $A_{1,3}^4$ by $A_{1,2}^4$.

2.1 The inverse of function of many variables

Definition 2.3 The inverse of function of many variables

Let $\varphi^{\tilde{n}}(x_1, x_2, \dots, x_i, \dots, x_n)$ is a function of n variables. $\xi_i^{\tilde{1}} : x_i \mapsto \varphi^{\tilde{n}}(x_1, x_2, \dots, x_i, \dots, x_n)$. $\varphi^{\tilde{n}}$ is a invertible function of many variables about x_i if $\xi_i^{\tilde{1}}$ is bijection for any $x_j (j = 1, 2, \dots, n, j \neq i)$.

For example, $f(x_1, x_2) = x_1^3 + x_2^2$ is invertible about variable x_1 and is not invertible about variable x_2 .

we define $(\varphi^{\tilde{n}})^{-i}$, the i-th inverse of $\varphi^{\tilde{n}}$ as:

$$x_i = (\varphi^{\tilde{n}})^{-i}(x_1, \dots, x_{i-1}, x_0, x_{i+1}, \dots, x_m) \quad (5)$$

To express the relation between $\varphi^{\tilde{n}}$ and $(\varphi^{\tilde{n}})^{-i}$ we introduce inverse operator I_i

$$(\varphi^{\tilde{n}})^{-i} = I_i(\varphi^{\tilde{n}}) \quad (6)$$

Definition 2.4 The piecewise invertible function of many variables

$$x_0 = \varphi^{\tilde{n}}(x_1, \dots, x_i, \dots, x_n),$$

A non-reversible function of n variables $\varphi^{\tilde{n}}$ defined on domain D is called a piecewise invertible function of many variables if we can divide D into several sub-domains properly, $d_1 \cup d_2, \dots, d_j, \dots, \cup d_m = D, d_1 \cap d_2, \dots, d_j, \dots, \cap d_m = \emptyset$ and $\varphi^{\tilde{n}}$ is invertible in each $d_j, j = 1, \dots, m$.

We consider each piece of $\varphi^{\tilde{n}}$ being invertible on d_j as a function of n variables, $\varphi_j^{\tilde{n}}$.

3. Application of the composite and Inverse for multivariate functions

Let us solve an equation which contains parameterized functions:

Replace the parameters u_1, u_2, u_3, u_4 in the expression $\left(\{[A_{1,3}^4(\varphi_3)]C_1^4[A_{1,2}^4(\varphi_1)]\}C_3^4[A_{3,4}^4(\varphi_2)]\right)(u_1, u_2, u_3, u_4)$ mentioned above by x,a,x,b respectively then we obtain equation:

$$\left(\{[A_{1,3}^4(\varphi_3)]C_1^4[A_{1,2}^4(\varphi_1)]\}C_3^4[A_{3,4}^4(\varphi_2)]\right)(x, a, x, b) = c$$

By the oblique projection of $\{[A_{1,3}^4(\varphi_3)]C_1^4[A_{1,2}^4(\varphi_1)]\}C_3^4[A_{3,4}^4(\varphi_2)]\}$ we have:

$$\left[C_{1,3}^4 \left(\{[A_{1,3}^4(\varphi_3)]C_1^4[A_{1,2}^4(\varphi_1)]\}C_3^4[A_{3,4}^4(\varphi_2)] \right) \right] (x, a, b) = c$$

Here we amuse $C_{1,3}^4 \left(\{[A_{1,3}^4(\varphi_3)]C_1^4[A_{1,2}^4(\varphi_1)]\}C_3^4[A_{3,4}^4(\varphi_2)] \right)$ is invertible or piecewise invertible. By the inverse of $C_{1,3}^4 \left(\{[A_{1,3}^4(\varphi_3)]C_1^4[A_{1,2}^4(\varphi_1)]\}C_3^4[A_{3,4}^4(\varphi_2)] \right)$ we obtain the solution finally:

$$x = \left\{ I_1 \left[C_{1,3}^4 \left(\{[A_{1,3}^4(\varphi_3)]C_1^4[A_{1,2}^4(\varphi_1)]\}C_3^4[A_{3,4}^4(\varphi_2)] \right) \right] \right\} (c, a, b)$$

4. Algebraic System of Equations

An algebraic system of equations (B,E) is constructed by a non-empty set B and a set E of equations constructed by the binary functions defined on B.

If elements of B are numbers then equations in (B,E) are algebraic equations. If elements of B are functions then equations in (B,E) are operator equations. B can be set of complex numbers or of real numbers or of integers. Algebraic systems of equations on set of complex numbers and on set of real numbers will be our main target. If B is a finite set then binary functions defined on it will be discrete ones.

5. Superposition Theorem

Theorem 3.4 Kolmogorov's Superposition Theorem Let $f^{\tilde{n}}: [0, 1]^n \rightarrow R$ be an arbitrary multivariate continuous function. Then it has the representation.

$$f^{\tilde{n}}(x_1, x_2, \dots, x_n) = \sum_{q=0}^{2n} \Phi_q^{\tilde{1}} \left[\sum_{p=1}^n \psi_{q,p}^{\tilde{1}}(x_p) \right] \quad (7)$$

Here a set of inner functions $\psi_{q,p}^{\tilde{1}}$ is not unique but any selected set will be independent to $f^{\tilde{n}}$ then they are called remaining function.

Most of the results about Hilbert 13th problem are existence but there are some constructive ones. We selected one set of $\psi_{q,p}^{\tilde{1}}$ from Jürgen Braun and Michael Griebel[3] and then the relation between $\varphi_q^{\tilde{1}}$ and $f^{\tilde{n}}$ will be given like below:

Definition 5.1 Decomposite operator For expressing the relation between $\varphi_q^{\tilde{1}}$ and $f^{\tilde{n}}$, we introduce decomposite operator D_q

$$\varphi_q^{\bar{1}} = D_q(f^{\bar{n}}), (0 \leq q \leq 2n) \quad (8)$$

It can be expressed too:

$$f^{\bar{n}}(x_1, x_2, \dots, x_n) = \sum_{q=0}^{2n} D_q(f^{\bar{n}}) \left[\sum_{p=1}^n \psi_{q,p}^{\bar{1}}(x_p) \right] \quad (9)$$

First A.N.Kolmogorov give a conclusion that any multivariate functions can be expressed by functions of three variables[1]. Further more V.I.Arnold proved that any functions of three variables can be expressed by binary functions[4],[5]. By this we can transfer the solutions expressed by multivariate functions to ones expressed by binary functions. By formula 8 we can obtain solutions expressed in unary function. Note, if we replaced x by $(y-a)/(b-a)$ then $[a,b]$ will become $[0,1]$ so formula 6 is suited to general domain.

Unary functions in Kolmogorov's superposition theorem are continues. Hilberts 13th problem is open for smooth situation. We are looking forward the positive result for it. Superposition theorem can be extended to multivariate functions defined on finite set. Some results about it will be shown in another paper.

6. Discussion and expectation

Structural mathematics, topology and algebra have been the main role of mathematics for more than one hundred years since Hilbert's solving Gordan problem. Meanwhile the method of classical mathematics is nearly forgotten by mathematicians. But resources in topology and algebra will be dried up in the near future and mathematics must go back to the path of classical mathematics as there are so mineral in it.

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